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Twistorial construction of generalized Kähler manifolds

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Abstract

The twistor method is applied for obtaining examples of generalized Kähler structures which are not yielded by Kähler structures.

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1. Introduction

The theory of generalized complex structures has been initiated by Hitchin [12] and further developed by Gualtieri [11]. These structures contain the complex and symplectic structures as special cases and can be considered as a complex analog of the notion of a Dirac structure introduced by Courant and Weinstein [6,7] to unify the Poisson and presymplectic geometries. This and the fact that the target spaces of supersymmetric σ-models are generalized complex manifolds motivate the increasing interest in generalized complex geometry.

The idea of this geometry is to replace the tangent bundle TM of a smooth manifold M with the bundle $TM \oplus T^*M$ endowed with the indefinite metric $\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X))$, $X, Y \in TM$, $\xi, \eta \in T^*M$. A generalized Kähler structure is, by definition, a pair $\{J_1, J_2\}$ of commuting generalized complex structures such that the quadratic form $\langle J_1 A, J_2 A\rangle$ is positive definite on $TM \oplus T^*M$. According to a result of Gualtieri [11] the generalized Kähler structures have an equivalent interpretation in terms of the so-called bi-Hermitian structures.

Any Kähler structure yields a generalized Kähler structure in a natural way. Non-trivial examples of such structures can be found in [2,3,5,13–16]. The purpose of the present paper is to provide non-trivial examples of generalized Kähler manifolds by means of the Penrose [17] twistor construction as developed by Atiyah, Hitchin and Singer [4] in the framework of Riemannian geometry.

Let M be a 2-dimensional smooth manifold. Following the general scheme of the twistor construction we consider the bundle $\mathcal{P}$ over M whose fibre at a point $p \in M$ consists of all pairs of commuting generalized complex structures $\{I, J\}$ on the vector space T_pM such that the form $\langle IA, JA\rangle$ is positive definite on $T_pM \oplus T_p^*M$. The general fibre

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of $\mathcal{P}$ admits two natural Kähler structures (in the usual sense) and can be identified in a natural way with the disjoint union of two copies of the unit bi-disk. Under this identification, the two structures are defined on the unit bi-disk as $(h \times h, \mathcal{K} \times (\pm\mathcal{K}))$ where h is the Poincare metric on the unit disk and $\mathcal{K}$ is its standard complex structure. These two Kähler structures yield a generalized Kähler structure on the fibre of $\mathcal{P}$ according to the Gualtieri result mentioned above. Moreover, any linear connection ∇ on M gives rise to a splitting of the tangent bundle $T\mathcal{P}$ into horizontal and vertical parts and this allows one to define two commuting generalized almost complex structures $\mathcal{I}^{\nabla}$ and $\mathcal{J}^{\nabla}$ on $\mathcal{P}$ such that the form $\langle \mathcal{I}^{\nabla}\cdot, \mathcal{J}^{\nabla}\cdot\rangle$ is positive definite on $T\mathcal{P} \oplus T^*\mathcal{P}$. The main result of the paper states that if the connection ∇ is torsion-free, the structures $\mathcal{I}^{\nabla}$ and $\mathcal{J}^{\nabla}$ are both integrable if and only if ∇ is flat. Thus any affine structure on M yields a generalized Kähler structure on the 6-dimensional manifold $\mathcal{P}$. Note that the only complete affine 2-dimensional manifolds are the plane, a cylinder, a Klein bottle, a torus, or a Mobius band [10,9].

2. Generalized Kähler structures

Let W be a n-dimensional real vector space and g a metric of signature (p, q) on it, $p + q = n$. We shall say that a basis $\{e_1, \dots, e_n\}$ of W is *orthonormal* if $\|e_1\|^2 = \dots = \|e_p\|^2 = 1$, $\|e_{p+1}\|^2 = \dots = \|e_{p+q}\|^2 = -1$. If $n = 2m$ is an even number and $p = q = m$, the metric g is usually called *neutral*. Recall that a complex structure J on W is called *compatible* with the metric g, if the endomorphism J is g-skew-symmetric.

Suppose that $\dim W = 2m$ and g is of signature $(2p, 2q)$, $p + q = m$. Denote by $J(W)$ the set of all complex structures on W compatible with the metric g. The group $O(g)$ of orthogonal transformations of W acts transitively on $J(W)$ by conjugation and $J(W)$ can be identified with the homogeneous space $O(2p, 2q)/U(p, q)$. In particular, $\dim J(W) = m^2 - m$. The group $O(2p, 2q)$ has four connected components, while $U(p, q)$ is connected, therefore $J(W)$ has four components.

Example 1 (*[8]*). The space $O(2, 2)/U(1, 1)$ is the disjoint union of two copies of the hyperboloid $x_1^2 - x_2^2 - x_3^2 = 1$.

Consider $J(W)$ as a (closed) submanifold of the vector space $so(g)$ of g-skew-symmetric endomorphisms of W. Then the tangent space of $J(W)$ at a point J consists of all endomorphisms $Q \in so(g)$ anti-commuting with J. Thus we have a natural $O(g)$-invariant almost complex structure $\mathcal{K}$ on $J(W)$ defined by $\mathcal{K}Q = J \circ Q$. It is easy to check that this structure is integrable.

Fix an orientation on W and denote by $J^{\pm}(W)$ the set of compatible complex structures on W that induce $\pm$ the orientation of W. The set $J^{\pm}(W)$ has the homogeneous representation $SO(2p, 2q)/U(p, q)$ and, thus, is the union of two components of $J(W)$.

Suppose that $\dim W = 4$ and g is of split signature (2, 2). Let $g(a, b) = -\frac{1}{2}$ Trace $(a \circ b)$ be the standard metric of $so(g)$. The restriction of this metric to the tangent space T_J of $J(W)$ is negative definite and we set $h = -g$ on T_J. Then the complex structure $\mathcal{K}$ is compatible with the metric h and $(\mathcal{K}, h)$ is a Kähler structure on $J(W)$. The space $J^{\pm}(W)$ can be identified with the hyperboloid $x_1^2 - x_2^2 - x_3^2 = 1$ in $\mathbb{R}^3$ (see e.g. [8, Example 5]) and it is easy to check that, under this identification, the structure $(\mathcal{K}, h)$ on $J^{\pm}(W)$ goes to the standard Kähler structure of the hyperboloid. Thus the Hermitian manifold $(J^{\pm}(W), \mathcal{K}, h)$ is biholomorphically isometric to the disjoint union of two copies of the unit disk endowed with the Poincare–Bergman metric (of curvature -1).

Let $\flat : T_J \to T_J^*$ and $\sharp = \flat^{-1}$ be the "musical" isomorphisms determined by the metric h. Denote by $T_J^{\perp}$ the orthogonal complement of T_J in $so(g)$ with respect to the metric g; the space $T_J^{\perp}$ consists of the skew-symmetric endomorphisms of W commuting with J. Consider T_J^* as the space of linear forms on $so(g)$ vanishing on $T_J^{\perp}$. Then for every $U \in T_J$ and $\omega \in T_J^*$ we have $U^{\flat}(A) = -g(U, A)$ and $g(\omega^{\sharp}, A) = -\omega(A)$ for every $A \in so(g)$.

Now let V be a real vector space and V^* its dual space. Then the vector space $V \oplus V^*$ admits a natural neutral metric defined by

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X)). \tag{1}$$

A *generalized complex structure* on the vector space V is, by definition, a complex structure on the space $V \oplus V^*$ compatible with its natural neutral metric [12]. If a vector space V admits a generalized complex structure, it is necessarily of even dimension [11]. We refer to [11] for more facts about the generalized complex structures.

Example 2 ([11–13]). Every complex structure K and every symplectic form ω on V (i.e. a non-degenerate 2-form) induce generalized complex structures on V in a natural way. If we denote these structures by J and S, respectively, the structure J is defined by $J = K$ on V and $J = -K^*$ on V^*, where $(K^*\xi)(X) = \xi(KX)$ for $\xi \in V^*$ and $X \in V$.

The map $X \to \imath_X\,\omega$ (the interior product) is an isomorphism of V onto V^*. Denote this isomorphism also by ω. Then the structure S is defined by $S = \omega$ on V and $S = -\omega^{-1}$ on V^*.

Example 3 ([11–13]). Any 2-form $B \in \Lambda^2 V^*$ acts on $V \oplus V^*$ via the inclusion $\Lambda^2 V^* \subset \Lambda^2(V \oplus V^*) \cong so(V \oplus V^*)$; in fact this is the action $X + \xi \to \imath_X\, B$; $X \in V, \xi \in V^*$. Denote the latter map again by B. Then the invertible map e^B is given by $X + \xi \to X + \xi + \imath_X\, B$ and is an orthogonal transformation of $V \oplus V^*$. Thus, given a generalized complex structure J on V, the map $\mathrm{e}^B J \mathrm{e}^{-B}$ is also a generalized complex structure on V, called the B-transform of J.

Similarly, any 2-vector $\beta \in \Lambda^2 V$ acts on $V \oplus V^*$. If we identify V with $(V^*)^*$, so $\Lambda^2 V \cong \Lambda^2(V^*)^*$, the action is given by $X + \xi \to \imath_\xi\, \beta \in V$. Denote this map by β. Then the exponential map e^β acts on $V \oplus V^*$ via $X + \xi \to X + \imath_\xi\, \beta + \xi$, in particular e^β is an orthogonal transformation. Hence, if J is a generalized complex structure on V, so is $\mathrm{e}^\beta J \mathrm{e}^{-\beta}$. It is called the β-transform of J.

Let $\{e_i\}$ be an arbitrary basis of V and $\{\eta_i\}$ its dual basis, $i = 1, \ldots, 2n$. Then the orientation of the space $V \oplus V^*$ determined by the basis $\{e_i, \eta_i\}$ does not depend on the choice of the basis $\{e_i\}$. Further on, we shall always consider $V \oplus V^*$ with this *canonical orientation*. The sets $J^\pm(V \oplus V^*)$ of generalized complex structures on V inducing $\pm$ the canonical orientation of $V \oplus V^*$ will be denoted by $G^\pm(V)$.

Example 4. A generalized complex structure on V induced by a complex structure (see Example 2) always yields the canonical orientation of $V \oplus V^*$. A generalized complex structure on V induced by a symplectic form yields the canonical orientation of $V \oplus V^*$ if and only if $n = \frac{1}{2}\dim V$ is an even number. The B- or β-transform of a generalized complex structure J on V yields the canonical orientation of $V \oplus V^*$ if and only if J does so.

Example 5. Let V be a 2-dimensional real vector space. Take a basis $\{e_1, e_2\}$ of V and let $\{\eta_1, \eta_2\}$ be its dual basis. Then $\{Q_1 = e_1 + \eta_1, Q_2 = e_2 + \eta_2, Q_3 = e_1 - \eta_1, Q_4 = e_2 - \eta_2\}$ is an orthonormal basis of $V \oplus V^*$ with respect to the natural neutral metric (1) and is positively oriented with respect to the canonical orientation of $V \oplus V^*$. Put $\varepsilon_k = \|Q_k\|^2, k = 1, \ldots, 4$, and define skew-symmetric endomorphisms of $V \oplus V^*$ setting $S_{ij} Q_k = \varepsilon_k(\delta_{ik} Q_j - \delta_{kj} Q_i)$, $1 \le i, j, k \le 4$. Then the endomorphisms

$$\begin{aligned} I_1 &= S_{12} - S_{34}, & J_1 &= S_{12} + S_{34}, \\ I_2 &= S_{13} - S_{24}, & J_2 &= S_{13} + S_{24}, \\ I_3 &= S_{14} + S_{23}, & J_3 &= S_{14} - S_{23} \end{aligned}$$

constitute a basis of the space of skew-symmetric endomorphisms of $V \oplus V^*$. Let $I \in G^+(V)$ and $J \in G^-(V)$. Then $I = \sum_r x_r I_r$ with $x_1^2 - x_2^2 - x_3^2 = 1$ and $J = \sum_s y_s J_s$ with $y_1^2 - y_2^2 - y_3^2 = 1$. It follows that

$$\begin{aligned} I e_1 &= x_2 e_1 + (x_1 + x_3) e_2, & J e_1 &= y_2 e_1 + (y_1 - y_3)\eta_2, \\ I e_2 &= -(x_1 - x_3) e_1 - x_2 e_2, & J e_2 &= y_2 e_2 - (y_1 - y_3)\eta_1, \\ I \eta_1 &= -x_2 \eta_1 + (x_1 - x_3)\eta_2, & J \eta_1 &= (y_1 + y_3) e_2 - y_2 \eta_1, \\ I \eta_2 &= -(x_1 + x_3)\eta_1 + x_2 \eta_2, & J \eta_2 &= -(y_1 + y_3) e_1 - y_2 \eta_2. \end{aligned}$$

This shows that the restriction of I to V is a complex structure on V inducing the generalized complex structure I (as in Example 2). In contrast, the generalized complex structure J is not induced by a complex structure or a symplectic form on V. Moreover J is not a B- or β-transform of such structures.

A generalized almost complex structure on an even-dimensional smooth manifold M is, by definition, an endomorphism J of the bundle $TM \oplus T^*M$ with $J^2 = -Id$ which preserves the natural neutral metric of $TM \oplus T^*M$. Such a structure is said to be *integrable* or *a generalized complex structure* if its $+i$-eigensubbundle of $(TM \oplus T^*M) \otimes \mathbb{C}$ is closed under the Courant bracket [12]. Recall that if X, Y are vector fields on M and ξ, η are 1-forms, the Courant bracket [6] is defined by the formula

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\imath_X\, \eta - \imath_Y\, \xi),$$

where $[X, Y]$ on the right hand-side is the Lie bracket and $\mathcal{L}$ means the Lie derivative. As in the case of almost complex structures, the integrability condition for a generalized almost complex structure J is equivalent to the vanishing of its Nijenhuis tensor N, the latter being defined by means of the Courant bracket:

$$N(A, B) = -[A, B] - J[A, JB] - J[JA, B] + [JA, JB], A, B \in TM \oplus T^*M.$$

Example 6 (*[11]*). A generalized complex structure K induced by an almost complex structure K on M (see Example 2) is integrable if and only if the structure K is integrable. A generalized complex structure yielded by a non-degenerate 2-form ω on M is integrable if and only if the form ω is closed.

Example 7 (*[11]*). Let J be a generalized almost complex structure and B a closed 2-form on M. Then the B-transform of J, $e^B J e^{-B}$, (see Example 3) is integrable if and only if the structure J is integrable.

Let us note that the notion of B-transform plays an important role in the local description of the generalized complex structures given by Gualtieri [11] and Abouzaid–Boyarchenko [1].

The existence of a generalized almost complex structure on a $2n$-dimensional manifold M is equivalent to the existence of a reduction of the structure group of the bundle $TM \oplus T^*M$ to the group $U(n, n)$. Further, to reduce the structure group to the subgroup $U(n) \times U(n)$ of $U(n, n)$ is equivalent to choosing two commuting generalized almost complex structures $\{J_1, J_2\}$ such that the quadratic form $\langle J_1 A, J_2 A\rangle$ on $TM \oplus T^*M$ is positive definite [11]. A pair $\{J_1, J_2\}$ of generalized complex structures with these properties is called an *almost generalized Kähler structure*. It is said to be a *generalized Kähler structure* if J_1 and J_2 are both integrable [11].

Example 8 (*[11]*). Let (J, g) be a Kähler structure on a manifold M and ω its Kähler form, $\omega(X, Y) = g(JX, Y)$. Let J_1 and J_2 be the generalized complex structures on M induced by J and ω. Then the pair $\{J_1, J_2\}$ is a generalized Kähler structure.

Example 9 (*[11]*). If $\{J_1, J_2\}$ is a generalized Kähler structure and B is a closed 2-form, then its B-transform $\{e^B J_1 e^{-B}, e^B J_2 e^{-B}\}$ is also a generalized Kähler structure.

It has been observed by Gualtieri [11] that an almost generalized Kähler structure $\{J_1, J_2\}$ on a manifold M determines the following data on M: (1) a Riemannian metric g; (2) two almost complex structures $J_\pm$ compatible with g; (3) a 2-form b. Conversely, the almost generalized Kähler structure $\{J_1, J_2\}$ can be reconstructed from the data (g, J_+, J_-, b). In fact, Gualtieri [11] has given an explicit formula for J_1 and J_2 in terms of this data.

Example 10. Let V be a 2-dimensional real vector space and $G^\pm(V)$ the space of generalized complex structures on V yielding $\pm$ the canonical orientation of $V \oplus V^*$. Let $(h, \mathcal{K})$ be the Kähler structure on $G^\pm(V)$ defined above. Consider the manifold $G^+(V) \times G^-(V)$ with the product metric $g = h \times h$ and the complex structures $J_+ = \mathcal{K} \times \mathcal{K}$ and $J_- = \mathcal{K} \times (-\mathcal{K})$. According to [11, formula (6.3)] the generalized Kähler structure $\{\mathcal{I}, \mathcal{J}\}$ on $G^+(V) \times G^-(V)$ determined by g, J_+, J_- and $b = 0$ is given by

$$\begin{aligned} &\mathcal{I}(U, V) = I \circ U - V^\flat \circ J, \qquad && \mathcal{J}(U, V) = J \circ V - U^\flat \circ I \\ &\mathcal{I}(\varphi, \psi) = -\varphi \circ I + J \circ \psi^\sharp, && \mathcal{J}(\varphi, \psi) = -\psi \circ J + I \circ \varphi^\sharp \end{aligned} \tag{2}$$

for $U \in T_I G^+(V)$, $V \in T_J G^-(V)$ and $\varphi \in T_I^* G^+(V)$, $\psi \in T_J^* G^-(V)$.

Gualtieri [11] has also proved that the integrability condition for $\{J_1, J_2\}$ can be expressed in terms of the data (g, J_+, J_-, b) in a nice way. In particular, in the case when $b = 0$, the structures $\{J_1, J_2\}$ are integrable if and only if the almost-Hermitian structures $(g, J_\pm)$ are Kählerian.

Example 11. According to the Gualtieri's result the structure $\{\mathcal{I}, \mathcal{J}\}$ defined by (2) is a generalized Kähler structure. Of course, the integrability of $\mathcal{I}$ and $\mathcal{J}$ can be directly proved.

Let V be an even-dimensional real vector space. The group $GL(V)$ acts on $V \oplus V^*$ by letting $GL(V)$ act on V^* in the standard way. This action preserves the neutral metric (1) and the canonical orientation of $V \oplus V^*$. Thus, we have an embedding of $GL(V)$ into the group $SO(\langle\ ,\ \rangle)$ and, via this embedding, $GL(V)$ acts on the manifold $G^\pm(V)$ in a natural manner. Denote by $P(V)$ the open subset of $G^+(V) \times G^-(V)$ consisting of those (I, J) for which the quadratic form $\langle IA, JA\rangle$ is positive definite on $V \oplus V^*$. It is clear that the natural action of $GL(V)$ on $G^+(V) \times G^-(V)$ leaves $P(V)$ invariant. Suppose that $\dim V = 2$. Let $I \in G^+(V)$ and $J \in G^-(V)$. Then it is easy to see that, under the notations in Example 5, the quadratic form $\langle IA, JA\rangle$ is positive definite if and only if either $x_1 + x_3 > 0$, $y_1 + y_3 > 0$ or $x_1 + x_3 < 0$, $y_1 + y_3 < 0$. This is equivalent to the condition that either $x_1 > 0$, $y_1 > 0$ or $x_1 < 0$, $y_1 < 0$. Thus $P(V)$ is the disjoint union of two products of one-sheeted hyperboloids. Therefore $P(V)$ endowed with the complex structure $\mathcal{K} \times \mathcal{K}$ and the metric $h \times h$ is biholomorphically isometric to the disjoint union of two copies of the unit bi-disk endowed with the Bergman metric. Note also that, when $\dim V = 2$, every $I \in G^+(V)$ *commutes* with every $J \in G^-(V)$ (see Example 5). Thus, in this case, every pair $(I, J) \in P(V)$ is a generalized Kähler structure on the manifold V.

3. The twistor space of generalized Kähler structures

Let M be a smooth manifold of dimension 2. Denote by $\pi : \mathcal{G}^{\pm} \to M$ the bundle over M whose fibre at a point $p \in M$ consists of all generalized complex structures on T_pM that induce $\pm$ the canonical orientation of $T_pM \oplus T_p^*M$. This is the associated bundle

$$GL(M) \times_{GL(2,\mathbb{R})} G^{\pm}(\mathbb{R}^2),$$

where $GL(M)$ denotes the principal bundle of linear frames on M. Consider the product bundle $\pi : \mathcal{G}^+ \times \mathcal{G}^- \to M$ and denote by $\mathcal{P}$ its open subset consisting of those pairs $K = (I, J)$ for which the quadratic form $\langle IA, JA\rangle$ on $T_pM \oplus T_p^*M$, $p = \pi(K)$, is positive definite. Clearly $\mathcal{P}$ is the associated bundle

$$\mathcal{P} = GL(M) \times_{GL(2,\mathbb{R})} P(\mathbb{R}^2).$$

The projection maps of the bundles $\mathcal{G}^{\pm}$ and $\mathcal{P}$ to the base space M will be denoted by π.

Let ∇ be a linear connection on M. Following the standard twistor construction we can define two commuting almost generalized complex structures $\mathcal{I}^{\nabla}$ and $\mathcal{J}^{\nabla}$ on $\mathcal{P}$ as follows: The connection ∇ gives rise to a splitting $\mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of any bundle associated to $GL(M)$ into vertical and horizontal parts. The vertical space $\mathcal{V}_K$ of $\mathcal{P}$ at a point $K = (I, J)$ is the direct sum $\mathcal{V}_K = \mathcal{V}_I\mathcal{G}^+ \oplus \mathcal{V}_J\mathcal{G}^-$ of vertical spaces and we define $\mathcal{I}^{\nabla}$ and $\mathcal{J}^{\nabla}$ on $\mathcal{V}_K$ by means of (2) where the "musical" isomorphisms are determined by the metric h on $\mathcal{V}_J\mathcal{G}^+$ and $\mathcal{V}_J\mathcal{G}^-$.

The horizontal space $\mathcal{H}_K$ is isomorphic via the differential π_{*K} to the tangent space T_pM, $p = \pi(K)$. Denoting $\pi_{*K}|\mathcal{H}$ by $\pi_{\mathcal{H}}$, we define $\mathcal{I}^{\nabla}$ and $\mathcal{J}^{\nabla}$ on $\mathcal{H}_K \oplus \mathcal{H}_K^*$ as the lift of the endomorphisms I and J by the map $\pi_{\mathcal{H}} \oplus (\pi_{\mathcal{H}}^{-1})^*$.

Remark. Neither of the generalized almost complex structures $\mathcal{I}^{\nabla}$ and $\mathcal{J}^{\nabla}$ is induced by an almost complex or symplectic structure on $\mathcal{P}$. Moreover they are not B- or β-transforms of such structures.

Further on the generalized almost complex structures $\mathcal{I}^{\nabla}$ and $\mathcal{J}^{\nabla}$ will be simply denoted by $\mathcal{I}$ and $\mathcal{J}$ when the connection ∇ is understood. The image of every $A \in T_pM \oplus T_p^*M$ under the map $\pi_{\mathcal{H}}^{-1} \oplus \pi_{\mathcal{H}}^*$ will be denoted by A^h. The elements of $\mathcal{H}_J^*$, resp. $\mathcal{V}_J^*$, will be considered as 1-forms on $T_J\mathcal{G}$ vanishing on $\mathcal{V}_J$, resp. $\mathcal{H}_J$.

Let $K = (I, J) \in \mathcal{P}$, $A \in T_{\pi(K)}M \oplus T_{\pi(K)}^*M$, $W = (U, V) \in \mathcal{V}_K$ and $\Theta = (\varphi, \psi) \in \mathcal{V}_K^*$. Then we have

$$\langle \mathcal{I}(A^h + W + \Theta), \mathcal{J}(A^h + W + \Theta)\rangle = \langle IA, JA\rangle + \|U\|_h^2 + \|V\|_h^2 + \|\varphi\|_h^2 + \|\psi\|_h^2.$$

Therefore the quadratic form $\langle \mathcal{I}\cdot, \mathcal{J}\cdot\rangle$ is positive definite. Thus the pair $(\mathcal{I}, \mathcal{J})$ is an almost generalized Kähler structure.

We shall show that for a torsion-free connection ∇ the integrability condition for $\mathcal{I}$ and $\mathcal{J}$ can be expressed in terms of the curvature of ∇ (as is usual in the twistor theory).

Let $A(M)$ be the bundle of the endomorphisms of $TM \oplus T^*M$ which are skew-symmetric with respect to its natural neutral metric $\langle\ ,\ \rangle$; the fibre of this bundle at a point $p \in M$ will be denoted by $A_p(M)$. The connection ∇ on TM induces a connection on $A(M)$, thus a connection on the bundle $A(M) \oplus A(M)$, both denoted again by ∇.

Consider the bundle $\mathcal{P}$ as a subbundle of the bundle $\pi : A(M) \oplus A(M) \to M$. Then the inclusion of $\mathcal{P}$ is fibre-preserving and the horizontal space of $\mathcal{P}$ at a point K coincides with the horizontal space of $A(M) \oplus A(M)$ at that point since the inclusion $P(\mathbb{R}^2) \subset so(2,2) \times so(2,2)$ is $SO(2,2)$-equivariant.

Let (U, x_1, x_2) be a local coordinate system of M and $\{Q_1, \dots, Q_4\}$ an orthonormal frame of $TM \oplus T^*M$ on U. Set $\varepsilon_k = \|Q_k\|^2$, $k = 1, \dots, 4$, and define sections S_{ij}, $1 \le i, j \le 4$, of $A(M)$ by the formula

$$S_{ij} Q_k = \varepsilon_k(\delta_{ik} Q_j - \delta_{kj} Q_i). \tag{3}$$

Then S_{ij}, $i < j$, form an orthogonal frame of $A(M)$ with respect to the metric $\langle a, b\rangle = -\frac{1}{2}\,\mathrm{Trace}\,(a \circ b)$; $a, b \in A(M)$; moreover $\|S_{ij}\|^2 = \varepsilon_i\varepsilon_j$ for $i \neq j$. For $c = (a, b) \in A(M) \oplus A(M)$, we set

$$\tilde{x}_m(c) = x_m \circ \pi(c), \qquad y_{ij}(c) = \varepsilon_i\varepsilon_j\langle a, S_{ij}\rangle, \qquad z_{ij}(c) = \varepsilon_i\varepsilon_j\langle b, S_{ij}\rangle.$$

Then $(\tilde{x}_m, y_{ij}, z_{kl})$, $m = 1, 2$, $1 \le i < j \le 4$, $1 \le k < l \le 4$, is a local coordinate system on the total space of the bundle $A(M) \oplus A(M)$. Note that $(\tilde{x}_m, y_{ij})$ and $(\tilde{x}_m, z_{kl})$ are local coordinate systems of the manifold $A(M)$.

Let

$$U = \sum_{i<j} u_{ij}\frac{\partial}{\partial y_{ij}}(I), \qquad V = \sum_{i<j} v_{ij}\frac{\partial}{\partial z_{ij}}(J)$$

be vertical vectors of $\mathcal{G}^+$ and $\mathcal{G}^-$ at some points I and J with $\pi(I) = \pi(J)$. It is convenient to set $u_{ij} = -u_{ji}$, $v_{ij} = -v_{ji}$ for $i \ge j$, $1 \le i, j \le 4$. Then the endomorphism U of $T_pM \oplus T_p^*M$, $p = \pi(I)$, is determined by $UQ_i = \sum_{j=1}^4 \varepsilon_i u_{ij} Q_j$; similarly for the endomorphism V of $T_pM \oplus T_p^*M$. Moreover

$$\mathcal{K}_I^* U^\flat = -(IU)^\flat = \sum_{i<j}\varepsilon_i\varepsilon_j\sum_{r=1}^4 u_{ir} y_{rj}(I)\varepsilon_r (dy_{ij})_I.$$

A similar formula holds for $\mathcal{K}_J^* V^\flat$. Thus we have

$$\mathcal{I}(U, V) = \sum_{i<j}\sum_r u_{ir} y_{rj}(I)\varepsilon_r\frac{\partial}{\partial y_{ij}}(I) - \sum_{k<l}\varepsilon_k\varepsilon_l\sum_s v_{ks} z_{sl}(J)\varepsilon_s (dz_{kl})_J \tag{4}$$

and

$$\mathcal{J}(U, V) = \sum_{k<l}\sum_s v_{ks} z_{sl}(J)\varepsilon_s\frac{\partial}{\partial z_{kl}}(J) - \sum_{i<j}\varepsilon_i\varepsilon_j\sum_r u_{ir} y_{rj}(I)\varepsilon_r (dy_{ij})_I. \tag{5}$$

Note also that, for every $A \in T_pM \oplus T_p^*M$, we have

$$A^h = \sum_{i=1}^{4n}(\langle A, Q_i\rangle \circ \pi)\varepsilon_i Q_i^h \tag{6}$$

and

$$\mathcal{I}A^h = \sum_{i,j=1}^4(\langle A, Q_i\rangle \circ \pi) y_{ij} Q_j^h, \qquad \mathcal{J}A^h = \sum_{k,l=1}^4(\langle A, Q_k\rangle \circ \pi) z_{kl} Q_l^h. \tag{7}$$

For each vector field

$$X = \sum_{i=1}^2 X^i\frac{\partial}{\partial x_i}$$

on U, the horizontal lift X^h on $\pi^{-1}(U)$ is given by

$$X^h = \sum_m (X^m \circ \pi)\frac{\partial}{\partial \tilde{x}_m} - \sum_{i<j}\sum_{a<b} y_{ab}(\langle \nabla_X S_{ab}, S_{ij}\rangle \circ \pi)\varepsilon_i\varepsilon_j\frac{\partial}{\partial y_{ij}}$$
$$- \sum_{k<l}\sum_{c<d} z_{cd}(\langle \nabla_X S_{cd}, S_{kl}\rangle \circ \pi)\varepsilon_k\varepsilon_l\frac{\partial}{\partial z_{kl}}. \tag{8}$$

Let $c = (a, b) \in A(M) \oplus A(M)$ and $p = \pi(c)$. Then (8) implies that, under the standard identification of $T_c(A_p(M) \oplus A_p(M))$ with the vector space $A_p(M) \oplus A_p(M)$, we have

$$[X^h, Y^h]_c = [X, Y]^h_c + R(X, Y)c, \tag{9}$$

where $R(X, Y)c = (R(X, Y)a, R(X, Y)b)$ is the curvature of the connection ∇ on $A(M) \oplus A(M)$ (for the curvature tensor we adopt the following definition: $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$).

Notation. Let $K = (I, J) \in \mathcal{P}$ and $p = \pi(K)$. There exists an oriented orthonormal basis $\{a_1, \ldots, a_4\}$ of $T_pM \oplus T^*_pM$ such that $a_2 = Ia_1$, $a_4 = Ia_3$ and $Ja_1 = \varepsilon a_2$, $Ja_3 = -\varepsilon a_4$ where, $\varepsilon = +1$ or -1. Let $\{Q_i\}$, $i = 1, \ldots, 4$, be an oriented orthonormal frame of $TM \oplus T^*M$ near the point p such that

$$Q_i(p) = a_i \quad \text{and} \quad \nabla Q_i|_p = 0, \quad i = 1, \ldots, 4.$$

Define sections S and T of $A(M)$ by setting

$$\begin{aligned} &SQ_1 = Q_2, \qquad SQ_2 = -Q_1, \qquad SQ_3 = Q_4, \qquad SQ_4 = -Q_3 \\ &TQ_1 = \varepsilon Q_2, \qquad JQ_2 = -\varepsilon Q_1, \qquad TQ_3 = -\varepsilon Q_4, \qquad TQ_4 = \varepsilon Q_3. \end{aligned}$$

Then $\nu = (S, T)$ is a section of $\mathcal{P}$ such that

$$\nu(p) = K, \qquad \nabla\nu|_p = 0$$

(considering ν as a section of $A(M) \oplus A(M)$). Thus $X^h_K = \nu_* X$ for every $X \in T_pM$.

Further, given a smooth manifold N, the natural projections of $TN \oplus T^*N$ onto TN and T^*N will be denoted by π_1 and π_2, respectively.

We shall use the above notations throughout this section.

The next three technical lemmas can be easily proved by means of (7)–(9).

Lemma 1. *If A and B are sections of the bundle $TM \oplus T^*M$ near p, then:*

(i) $[\pi_1(A^h), \pi_1(\mathcal{I}B^h)]_K = [\pi_1(A), \pi_1(SB)]^h_K + R(\pi_1(A), \pi_1(IB))K$.
(ii) $[\pi_1(\mathcal{I}A^h), \pi_1(\mathcal{I}B^h)]_K = [\pi_1(SA), \pi_1(SB)]^h_K + R(\pi_1(IA), \pi_1(IB))K$.

Lemma 2. *Let A and B be sections of the bundle $TM \oplus T^*M$ near p, and let $Z \in T_pM$, $W = (U, V) \in \mathcal{V}_K = \mathcal{V}_I\mathcal{G}^+ \oplus \mathcal{V}_J\mathcal{G}^-$. Then:*

(i) $(\mathcal{L}_{\pi_1(A^h)}\pi_2(B^h))_K = (\mathcal{L}_{\pi_1(A)}\pi_2(B))^h_K$.
(ii) $(\mathcal{L}_{\pi_1(A^h)}\pi_2(\mathcal{I}B^h))_K = (\mathcal{L}_{\pi_1(A)}\pi_2(SB))^h_K$.
(iii) $(\mathcal{L}_{\pi_1(\mathcal{I}A^h)}\pi_2(B^h))_K(Z^h + W) = (\mathcal{L}_{\pi_1(SA)}\pi_2(B))^h_K(Z^h) + (\pi_2(B))_p(\pi_1(UA))$.
(iv) $(\mathcal{L}_{\pi_1(\mathcal{I}A^h)}\pi_2(\mathcal{I}B^h))_K(Z^h + W) = (\mathcal{L}_{\pi_1(SA)}\pi_2(SB))^h_K(Z^h) + (\pi_2(IB))_p(\pi_1(UA))$.

Lemma 3. *Let A and B are sections of the bundle $TM \oplus T^*M$ near p. Let $Z \in T_pM$ and $W = (U, V) \in \mathcal{V}_K = \mathcal{V}_I\mathcal{G}^+ \oplus \mathcal{V}_J\mathcal{G}^-$. Then:*

(i) $(d\,\iota_{\pi_1(A^h)}\,\pi_2(B^h))_K = (d\,\iota_{\pi_1(A)}\,\pi_2(B))^h_K$
(ii) $(d\,\iota_{\pi_1(A^h)}\,\pi_2(\mathcal{I}B^h))_K(Z^h + W) = (d\,\iota_{\pi_1(A)}\,\pi_2(SB))^h_K(Z^h) + (\pi_2(UB))_p(\pi_1(A))$
(iii) $(d\,\iota_{\pi_1(\mathcal{I}A^h)}\,\pi_2(B^h))_K(Z^h + W) = (d\,\iota_{\pi_1(SA)}\,\pi_2(B))^h_K(Z^h) + (\pi_2(B))_p(\pi_1(UA))$
(iv) $(d\,\iota_{\pi_1(\mathcal{I}A^h)}\,\pi_2(\mathcal{I}B^h))_K(Z^h + W) = (d\,\iota_{\pi_1(SA)}\,\pi_2(SB))^h_K(Z^h) + (\pi_2(UB))_p(\pi_1(IA)) + (\pi_2(IB))_p(\pi_1(UA))$.

Proposition 1. *Suppose that the connection ∇ is torsion-free and let $K = (I, J) \in \mathcal{P}$. Then*

(i) $N^{\mathcal{I}}(A^h, B^h) = 0$ *for every* $A, B \in T_{\pi(K)}M \oplus T^*_{\pi(K)}M$.
(ii) $N^{\mathcal{J}}(A^h, B^h) = 0$ *for every* $A, B \in T_{\pi(K)}M \oplus T^*_{\pi(K)}M$ *if and only if* $R(X, Y)J = 0$ *for every* $X, Y \in T_{\pi(K)}M$.

Proof. First we shall show that

$$\begin{aligned} N^{\mathcal{I}}(A^h, B^h)_K = & -R(\pi_1(A), \pi_1(B))I - I \circ R(\pi_1(A), \pi_1(IB))I \\ & - I \circ R(\pi_1(IA), \pi_1(B))I + R(\pi_1(IA), \pi_1(IB))I \\ & - R(\pi_1(A), \pi_1(B))J + R(\pi_1(IA), \pi_1(IB))J \\ & + \mathcal{K}^*_J(R(\pi_1(A), \pi_1(IB)J)^\flat) + \mathcal{K}^*_J(R(\pi_1(IA), \pi_1(B)J)^\flat). \end{aligned} \tag{10}$$

A similar formula holds for the Nijenhuis tensor $N^{\mathcal{J}}$ with interchanged roles of I and J in the right-hand side of (10).

Set $p = \pi(K)$ and extend A and B to (local) sections of $TM \oplus T^*M$, denoted again by A, B, in such a way that $\nabla A|_p = \nabla B|_p = 0$.

Let $\nu = (S, T)$ be the section of $\mathcal{P}$ defined above with the property that $\nu(p) = K$ and $\nabla\nu|_p = 0$ (ν being considered as a section of $A(M) \oplus A(M)$).

According to Lemmas 1–3, the part of $N^{\mathcal{I}}(A^h, B^h)_K$ lying in $\mathcal{H}_K \oplus \mathcal{H}^*_K$ is given by

$$(\mathcal{H} \oplus \mathcal{H}^*)N^{\mathcal{I}}(A^h, B^h)_K = (-[A, B] - S[A, SB] - S[SA, B] + [SA, SB])^h_K. \tag{11}$$

Note that we have $\nabla\pi_1(A)|_p = \pi_1(\nabla A|_p) = 0$ and $\nabla\pi_1(SA)|_p = \pi_1((\nabla S)|_p(A) + S(\nabla A|_p)) = 0$. Similarly, $\nabla\pi_2(A)|_p = 0$ and $\nabla\pi_2(SA)|_p = 0$. We also have $\nabla\pi_1(B)|_p = 0$, $\nabla\pi_1(SB)|_p = 0$ and $\nabla\pi_2(B)|_p = 0$, $\nabla\pi_2(SB)|_p = 0$. Now, since ∇ is torsion-free, we can easily see that every bracket in (11) vanishes by means of the following simple observation: Let Z be a vector field and ω a 1-form on M such that $\nabla Z|_p = 0$ and $\nabla\omega|_p = 0$. Then for every $T \in T_pM$

$$(\mathcal{L}_Z\omega)(T)_p = (\nabla_Z\omega)(T)_p = 0 \quad \text{and} \quad (d\, \imath_Z\, \omega)(T)_p = (\nabla_T\omega)(Z)_p = 0.$$

By Lemmas 1–3, the part of $N^{\mathcal{I}}(A^h, B^h)_K$ lying in $\mathcal{V}_K$ is

$$\begin{aligned} & -R(\pi_1(A), \pi_1(B))I - I \circ R(\pi_1(A), \pi_1(IB))I - I \circ R(\pi_1(IA), \pi_1(B))I + R(\pi_1(IA), \pi_1(IB))I \\ & -R(\pi_1(A), \pi_1(B))J + R(\pi_1(IA), \pi_1(IB))J. \end{aligned}$$

Finally, the part of $N^{\mathcal{I}}(A^h, B^h)_K$ lying in $\mathcal{V}^*_K$ is the vertical form whose value at every vertical vector $W = (U, V) \in \mathcal{V}_K$ is equal to

$$\begin{aligned} & \frac{1}{2}\{-\pi_2(IUB)(\pi_1(A)) - \pi_2(A)(\pi_1(IUB)) + \pi_2(IUA)(\pi_1(B)) + \pi_2(B)(\pi_1(IUA)) \\ & \quad + \pi_2(IB)(\pi_1(UA)) + \pi_2(UA)(\pi_1(IB)) - \pi_2(IA)(\pi_1(UB)) - \pi_2(UB)(\pi_1(IA))\} \\ & \quad + \mathcal{K}^*_J(R(\pi_1(A), \pi_1(IB)J)^\flat) + \mathcal{K}^*_J(R(\pi_1(IA), \pi_1(B)J)^\flat). \end{aligned}$$

The endomorphism U of $T_pM \oplus T^*_pM$ is skew-symmetric with respect to the metric $\langle\ ,\ \rangle$ and anti-commutes with I. Thus we have

$$\langle IUA, B\rangle = \langle IA, UB\rangle.$$

This identity reads as

$$\pi_2(IUA)(\pi_1(B)) + \pi_2(B)(\pi_1(IUA)) = \pi_2(IA)(\pi_1(UB)) + \pi_2(UB)(\pi_1(IA)).$$

Therefore the part of $N^{\mathcal{I}}(A^h, B^h)_K$ lying in $\mathcal{V}^*_K$ is

$$\mathcal{K}^*_J(R(\pi_1(A), \pi_1(IB)J)^\flat) + \mathcal{K}^*_J(R(\pi_1(IA), \pi_1(B)J)^\flat).$$

This proves formula (10).

Now let $\{Q_1, Q_2 = IQ_1, Q_3, Q_4 = IQ_3\}$ be an orthonormal basis of $T_pM \oplus T^*_pM$. To prove that $N^{\mathcal{I}}(A^h, B^h)_K = 0$ it is enough to show that $N^{\mathcal{I}}(Q^h_1, Q^h_3)_K = 0$ since $N^{\mathcal{I}}(\mathcal{I}E, F) = N^{\mathcal{I}}(E, \mathcal{I}F) = -\mathcal{I}N^{\mathcal{I}}(E, F)$ for every $E, F \in T\mathcal{P}$.

Let $\pi_1(Q_i) = e_i, i = 1, \ldots, 4$. Then, according to (10)

$$\begin{aligned} N^{\mathcal{I}}(Q^h_1, Q^h_3) = & [-R(e_1, e_3)I + R(e_2, e_4)I] - I \circ [R(e_1, e_4)I + R(e_2, e_3)I] \\ & + \mathcal{K}^*_J(R(e_1, e_4)J + R(e_2, e_3)J)^\flat. \end{aligned}$$

Since I yields the canonical orientation of $T_pM \oplus T_p^*M$, the latter expression vanishes in view of the following simple algebraic fact proved in [8]:

Lemma 4. *Let V be a 2-dimensional real vector space and let $\{Q_i = e_i + \eta_i\}$, $1 \le i \le 4$, be an orthonormal basis of the space $V \oplus V^*$ endowed with its natural neutral metric (1). Then $\{e_1, e_2\}$ is a basis of V and*

$$e_3 = a_{11}e_1 + a_{12}e_2$$
$$e_4 = a_{21}e_1 + a_{22}e_2$$

where $A = [a_{kl}]$ is an orthogonal matrix. If $\det A = 1$, the basis $\{Q_i\}$ yields the canonical orientation of $V \oplus V^*$ and if $\det A = -1$ it yields the opposite one.

To prove statement (ii), take an orthonormal basis $\{\bar{Q}_1, \bar{Q}_2 = J\bar{Q}_1, \bar{Q}_3, \bar{Q}_4 = J\bar{Q}_3\}$ and set $\pi_1(\bar{Q}_i) = e_i$, $i = 1, \ldots, 4$. Suppose that $N^{\mathcal{J}}(\bar{Q}_1^h, \bar{Q}_3^h) = 0$. Then, according to the analog of (10) for $N^{\mathcal{J}}(A^h, B^h)_K$, we have

$$-R(e_1, e_3)J + R(e_2, e_4)J - J \circ [R(e_1, e_4)J + R(e_2, e_3)J] = 0.$$

Since J yields the orientation of $T_pM \oplus T_p^*M$ opposite to the canonical one, then, by Lemma 4, $e_3 = \cos t\, e_1 + \sin t\, e_2$, $e_4 = \sin t\, e_1 - \cos t\, e_2$ for some $t \in \mathbb{R}$. Thus

$$-\sin t \cdot R(e_1, e_2)J + \cos t \cdot J \circ R(e_1, e_2)J = 0,$$

which implies

$$\cos t \cdot R(e_1, e_2)J + \sin t \cdot J \circ R(e_1, e_2)J = 0.$$

Therefore $R(e_1, e_2)J = 0$, so $R(X, Y)J = 0$ for every $X, Y \in T_pM$.

Conversely, if the latter identity holds, the analog of (10) shows that $N^{\mathcal{J}}(A^h, B^h)_K = 0$. □

Proposition 2. *Suppose that the connection ∇ is torsion-free and let $K = (I, J) \in \mathcal{P}$, Then*

(i) *$N^{\mathcal{I}}(A^h, W) = 0$ for every $A \in T_{\pi(K)}M \oplus T^*_{\pi(K)}M$ and $W \in \mathcal{V}_K$ if and only if $R(X, Y)J = 0$ for every $X, Y \in T_{\pi(K)}M$.*

(ii) *$N^{\mathcal{J}}(A^h, W) = 0$ for every $A \in T_{\pi(K)}M \oplus T^*_{\pi(K)}M$ and $W \in \mathcal{V}_K$ if and only if $R(X, Y)I = 0$ for every $X, Y \in T_{\pi(K)}M$.*

Proof. Set $p = \pi(K)$ and $W = (U, V)$. Extend A to a section of $TM \oplus T^*M$ denoted again by A. Take sections a and b of $A(M)$ such that

$$a(p) = U, \qquad b(p) = V, \qquad \nabla a|_p = \nabla b|_p = 0.$$

Define vertical vector fields $\widetilde{a}$ and $\widetilde{b}$ on $\mathcal{G}^+$ and $\mathcal{G}^-$, respectively, setting

$$\widetilde{a}_{I'} = a_{\pi(I')} + I' \circ a_{\pi(I')} \circ I', \quad I' \in \mathcal{G}^+ \quad \text{and} \quad \widetilde{b}_{J'} = b_{\pi(J')} + J' \circ b_{\pi(J')} \circ J', \quad J' \in \mathcal{G}^-. \tag{12}$$

Then

$$\widetilde{W}_{(I',J')} = (\widetilde{a}_{I'}, \widetilde{b}_{J'}), \quad (I', J') \in \mathcal{P},$$

is a vertical vector field on $\mathcal{P}$ with $\widetilde{W}_K = 2W$.

Let $a(Q_i) = \sum_j \varepsilon_i a_{ij} Q_j$, $b(Q_i) = \sum_j \varepsilon_i b_{ij} Q_j$. Then, in the local coordinates introduced above,

$$\widetilde{W} = \sum_{i<j} \left(\widetilde{a}_{ij} \frac{\partial}{\partial y_{ij}} + \widetilde{b}_{ij} \frac{\partial}{\partial z_{ij}} \right), \tag{13}$$

where

$$\widetilde{a}_{ij} = a_{ij} \circ \pi + \sum_{k,l} y_{ik}(a_{kl} \circ \pi) y_{lj} \varepsilon_k \varepsilon_l, \qquad \widetilde{b}_{ij} = b_{ij} \circ \pi + \sum_{k,l} z_{ik}(b_{kl} \circ \pi) z_{lj} \varepsilon_k \varepsilon_l.$$

In view of (8), for any vector field X on M near the point p, we have

$$X^h_K = \sum_m X^m(p)\frac{\partial}{\partial \tilde{x}_m}(K), \qquad \left[X^h, \frac{\partial}{\partial y_{ij}}\right]_K = \left[X^h, \frac{\partial}{\partial z_{ij}}\right]_K = 0, \tag{14}$$

and

$$0 = (\nabla_{X_p} a)(Q_i) = \sum_j \varepsilon_i X_p(a_{ij})Q_j, \qquad 0 = (\nabla_{X_p} b)(Q_i) = \sum_j \varepsilon_i X_p(b_{ij})Q_j$$

since $\nabla Q_i|_p = 0$ and $\nabla S_{ij}|_p = 0$. In particular, $X_p(a_{ij}) = X_p(b_{ij}) = 0$, hence

$$X^h_K(\widetilde{a}_{ij}) = X^h_K(\widetilde{b}_{ij}) = 0. \tag{15}$$

Now simple calculations making use of (14), (8) and (13) give

$$[X^h, \widetilde{W}]_K = 0. \tag{16}$$

Let ω be a 1-form on M. It is easy to see that for every vertical vector field W' on $\mathcal{P}$

$$[\omega^h, W'] = 0. \tag{17}$$

Therefore, by (16) and (17), we have

$$[A^h, \widetilde{W}]_K = 0. \tag{18}$$

Next, in view of (17), (4), (14) and (15), we have

$$[A^h, \mathcal{I}\widetilde{W}]_K = [\pi_1(A^h), \mathcal{I}\widetilde{W}]_K = (\mathcal{L}_{\pi_1(A^h)}\pi_2(\mathcal{I}\widetilde{W}))_K.$$

Let $W' = (U', V') \in \mathcal{V}_K$. Take sections a', b' of $A(M)$ such that $a'(p) = U'$, $b'(p) = V'$, $\nabla a'|_p = \nabla b'|_p = 0$. Define vertical vector fields $\widetilde{a'}$ and $\widetilde{b'}$ on $\mathcal{G}^+$ and $\mathcal{G}^-$ by means of (12) and set $\widetilde{W'} = (\widetilde{a'}, \widetilde{b'})$ on $\mathcal{P}$. Then $[X^h, \widetilde{W'}]_K = 0$ for every vector field X near the point p and an easy computation making use of (4), (14) and (15) gives

$$(\mathcal{L}_{\pi_1(A^h)}\pi_2(\mathcal{I}\widetilde{W}))_K(W') = \frac{1}{2}(\mathcal{L}_{\pi_1(A^h)}\pi_2(\mathcal{I}\widetilde{W}))_K(\widetilde{W'}) = 0.$$

Moreover, for every vector field Z on M near the point p we have

$$\begin{aligned}(\mathcal{L}_{\pi_1(A^h)}\pi_2(\mathcal{I}\widetilde{W}))_K(Z^h) &= -\pi_2(\mathcal{I}\widetilde{W})([\pi_1(A^h), Z^h]_K) \\ &= 2V^\flat(J \circ R(\pi_1(A), Z)J) = 2\langle JV, R(\pi_1(A), Z)J\rangle\end{aligned}$$

by (2) and (9). It is convenient to define a 1-form γ_A on T_pM setting

$$\gamma_A(Z) = \langle JV, R(\pi_1(A), Z)J\rangle, \quad Z \in T_pM.$$

Then

$$[A^h, \mathcal{I}\widetilde{W}]_K = 2\gamma^h_A.$$

Computations in local coordinates involving (7), (4), (14) and (15) show that

$$[\mathcal{I}A^h, \widetilde{W}]_K = -2(U(A))^h_K$$

and

$$[\mathcal{I}A^h, \mathcal{I}\widetilde{W}]_K = -2((IU)(A))^h_K + 2\gamma^h_{IA}.$$

It follows that

$$N^{\mathcal{I}}(A^h, W) = \frac{1}{2}N^{\mathcal{I}}(A^h, \widetilde{W})_K = -\mathcal{J}\gamma^h_A + \gamma^h_{IA}.$$

Let $\{e_1, e_2\}$ be a basis of T_pM and denote by $\{\eta_1, \eta_2\}$ its dual basis. Then $Q_1 = e_1 + \eta_1$, $Q_2 = e_2 + \eta_2$, $Q_3 = e_1 - \eta_1$, $Q_4 = e_2 - \eta_2$ constitute an orthonormal basis of $T_pM \oplus T_pM^*$ yielding its canonical orientation. According to Example 5, every generalized complex structure $J \in G^-(T_pM)$ is given by

$$\begin{aligned} &Q_1 \to y_1Q_2 + y_2Q_3 + y_3Q_4, &&Q_2 \to -y_1Q_1 + y_2Q_4 - y_3Q_3\\ &Q_3 \to -y_1Q_4 + y_2Q_1 - y_3Q_2, &&Q_4 \to y_1Q_3 + y_2Q_2 + y_3Q_1, \end{aligned}$$

where $y_1^2 - y_2^2 - y_3^2 = 1$, $y_1, y_2, y_3 \in \mathbb{R}$. Then

$$\begin{aligned} \mathcal{J}\gamma_A^h &= \gamma_A(e_1)(J\eta_1)^h + \gamma_A(e_2)(J\eta_2)^h\\ &= -(y_1 + y_3)\gamma_A(e_2)e_1^h + (y_1 + y_3)\gamma_A(e_1)e_2^h - y_2\gamma_A(e_1)\eta_1^h - y_2\gamma_A(e_2)\eta_2^h. \end{aligned}$$

Therefore the identity $N^{\mathcal{I}}(A^h, W) = 0$ implies $\gamma_A(e_1) = \gamma_A(e_2) = 0$, i.e. $\gamma_A = 0$. This proves statement (i). The proof of (ii) is similar. □

Now suppose that $R(X, Y)I = 0$ for every generalized complex structure $I \in G^+(T_pM)$, $X, Y \in T_pM$ being fixed. Take a basis $\{e_1, e_2\}$ of T_pM, denote by $\{\eta_1, \eta_2\}$ its dual basis and set $Q_1 = e_1 + \eta_1$, $Q_2 = e_2 + \eta_2$, $Q_3 = e_1 - \eta_1$, $Q_4 = e_2 - \eta_2$. Then every I is given by (see Example 5)

$$\begin{aligned} &Q_1 \to x_1Q_2 + x_2Q_3 + x_3Q_4, &&Q_2 \to -x_1Q_1 - x_2Q_4 + x_3Q_3\\ &Q_3 \to -x_1Q_4 + x_2Q_1 + x_3Q_2, &&Q_4 \to -x_1Q_3 - x_2Q_2 + x_3Q_1, \end{aligned}$$

where $x_1^2 - x_2^2 - x_3^2 = 1$, $x_1, x_2, x_3 \in \mathbb{R}$. The identity $R(X, Y)I = 0$ implies $\langle R(X, Y)Ie_1, \eta_k\rangle + \langle R(X, Y)e_1, I\eta_k\rangle = 0$, $k = 1, 2$, which is equivalent to

$$\begin{aligned} &(x_1 + x_3)\eta_1(R(X, Y)e_2) + (x_1 - x_3)\eta_2(R(X, Y)e_1) = 0,\\ &2x_2\eta_2(R(X, Y)e_2) - (x_1 + x_3)\eta_1(R(X, Y)e_1) + (x_1 + x_3)\eta_2(R(X, Y)e_2) = 0. \end{aligned}$$

It follows that $R(X, Y)I = 0$ for every I if and only if $R(X, Y) = 0$.

It is also easy to see that $R(X, Y)J = 0$ for every $J \in G^+(T_pM)$ if and only if $\eta_1(R(X, Y)e_1) + \eta_2(R(X, Y)e_2) = 0$.

Thus if the structures $\mathcal{I}$ and $\mathcal{J}$ are both integrable, then the connection ∇ is flat. The converse is also true as the following result shows.

Theorem 1. *Let M be a 2-dimensional manifold and ∇ a torsion-free connection on M. Then the generalized almost complex structures $\mathcal{I}$ and $\mathcal{J}$ induced by ∇ on the twistor space $\mathcal{P}$ are both integrable if and only if the connection ∇ is flat.*

Proof. Since the structures $\mathcal{I}$ and $\mathcal{J}$ on $\mathcal{V} \oplus \mathcal{V}^*$ are induced by complex structures on the fibres of $\mathcal{P}$ the Nijenhuis tensors of $\mathcal{I}$ and $\mathcal{J}$ vanish on $\mathcal{V} \oplus \mathcal{V}^*$. Thus, in view of Propositions 1 and 2, we have to consider these tensors only on $\mathcal{H} \times \mathcal{V}^*$.

Suppose that the connection ∇ is flat. Let $K = (I, J) \in \mathcal{P}$. Fix bases $\{U_1, U_2 = \mathcal{K}^+U_1\}$ of $\mathcal{V}_I\mathcal{G}^+$ and $\{V_1, V_2 = \mathcal{K}^-V_1\}$ of $\mathcal{V}_J\mathcal{G}^-$. Take sections a_1 and b_1 of $A(M)$ near the point $p = \pi(K)$ such that $a_1(p) = U_1$, $b_1(p) = V_1$ and $\nabla a_1|_p = \nabla b_1|_p = 0$. Define vertical vector fields $\widetilde{a_1}$ and $\widetilde{b_1}$ on $\mathcal{G}^+$ and $\mathcal{G}^-$ by means of (12). Set $\widetilde{a_2} = \mathcal{K}^+\widetilde{a_1}$, $\widetilde{b_2} = \mathcal{K}^-\widetilde{b_1}$. Then $\{\widetilde{a_1}, \widetilde{a_2}\}$ and $\{\widetilde{b_1}, \widetilde{b_2}\}$ are frames of the vertical bundles $\mathcal{V}\mathcal{G}^+$ and $\mathcal{V}\mathcal{G}^-$ near the points I and J, respectively. Denote by $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ the dual frames of $\{\widetilde{a_1}, \widetilde{a_2}\}$ and $\{\widetilde{b_1}, \widetilde{b_2}\}$. Set $\widetilde{W_i} = (\widetilde{a_i}, 0)$, $\gamma_i = (\alpha_i, 0)$ and $\widetilde{W_{i+2}} = (0, \widetilde{b_i})$, $\gamma_{i+2} = (0, \beta_i)$ for $i = 1, 2$. Then $\{\widetilde{W_1}, \widetilde{W_2}, \widetilde{W_3}, \widetilde{W_4}\}$ is a frame of the vertical bundle $\mathcal{V}$ of $\mathcal{P}$ near the point K and $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is its dual frame. We have $\gamma_2 = \mathcal{I}\gamma_1$, $\mathcal{I}\gamma_3 = \beta_2^\sharp$, $\mathcal{I}\gamma_4 = -\beta_1^\sharp$. If $A \in T_pM \oplus T_p^*M$, then $\mathcal{I}N^{\mathcal{I}}(A^h, \gamma_3) = -N^{\mathcal{I}}(A^h, \mathcal{I}\gamma_3) = -N^{\mathcal{I}}(A^h, \beta_2^\sharp) = 0$ by Proposition 2. Hence $N^{\mathcal{I}}(A^h, \gamma_3) = 0$. Similarly, $N^{\mathcal{I}}(A^h, \gamma_4) = 0$.

As in the proof of Proposition 2, it is not hard to see that $[\pi_1(A^h), \widetilde{W_r}]_K = 0$, $r = 1, \ldots, 4$, $[\pi_1(\mathcal{I}A^h), \widetilde{W_i}]_K = -(\pi_1(I\widetilde{a_i}(A)))_K^h$ and $[\pi_1(\mathcal{I}A^h), \widetilde{W_{i+2}}]_K = 0$, $i = 1, 2$. In particular $[\pi_1(A^h), \widetilde{W_r}]_K$ and $[\pi_1(\mathcal{I}A^h), \widetilde{W_r}]_K$ are horizontal vectors for every $r = 1, \ldots, 4$. It follows, in view of (9) and Lemma 1(i), that for every $Z \in T_pM$,

$r = 1, \dots, 4$ and $s = 1, 2$

$$(\mathcal{L}_{\pi_1(A^h)}\gamma_s)_K(Z^h + \widetilde{W}_r) = -\alpha_s(R(\pi_1(A), Z)I) = 0,$$
$$(\mathcal{L}_{\pi_1(\mathcal{I}A^h)}\gamma_s)_K(Z^h + \widetilde{W}_r) = -\alpha_s(R(\pi_1(IA), Z)I) = 0$$

since the connection ∇ is flat. This implies $N^{\mathcal{I}}(A^h, \gamma_s)_K = 0$ for $s = 1, 2$.

It follows that $N^{\mathcal{I}}(A^h, \Theta)_K = 0$ for every $\Theta \in \mathcal{V}_K^*$. Similarly, $N^{\mathcal{J}}(A^h, \Theta)_K = 0$. □

Denote by (g, J_+, J_-, b) the data on $\mathcal{P}$ determined by the almost generalized Kähler structure $\{\mathcal{I}^\nabla, \mathcal{J}^\nabla\}$ as described in [11]. It is not hard to see that the metric g, the almost complex structures $J_\pm$ and the 2-form b are given as follows. Let $K = (I, J) \in \mathcal{P}$, $X, Y \in T_{\pi(K)}M$, $W = (U, V) \in \mathcal{V}_K$. Let $\{e_1, e_2\}$ be a local frame of TM near the point $\pi(K)$ and denote by $\{\eta_1, \eta_2\}$ its dual co-frame. Define endomorphisms I_r, J_s, $r, s = 1, 2, 3$, by means of e_1, e_2, η_1, η_2 as in Example 5. Then $I = \sum_r x_r I_r$, $J = \sum_s y_s J_s$ with $x_1^2 - x_2^2 - x_3^2 = 1$, $y_1^2 - y_2^2 - y_3^2 = 1$. Let $X = X_1e_1 + X_2e_2$, $Y = Y_1e_1 + Y_2e_2$. Then

$$g(X^h, Y^h)_K = \frac{1}{y_1 + y_3}[(x_1 + x_3)X_1Y_1 - x_2(X_1Y_2 + X_2Y_1) + (x_1 - x_3)X_2Y_2],$$
$$g(X^h, W)_K = 0, \qquad g|(\mathcal{V}_K \times \mathcal{V}_K) = h.$$
$$J_+X_K^h = (IX)_K^h, \qquad J_-X_K^h = (IX)_K^h,$$
$$J_+(U, V) = (I \circ U, J \circ V), \qquad J_-(U, V) = (I \circ U, -J \circ V).$$
$$b(X^h, Y^h)_K = \frac{y_2}{y_1 + y_3}(X_1Y_2 - X_2Y_1),$$
$$b(X^h, W)_K = 0, \qquad b|(\mathcal{V}_K \times \mathcal{V}_K) = 0.$$

In particular, the almost complex structures J_+ and J_- commutes and $J_+ \neq \pm J_-$.

Computations similar to that above show that the almost complex structures $J_\pm$ are both integrable for any torsion-free connection ∇. Denote by $\omega_\pm$ the Kähler form of the Hermitian structure $(g, J_\pm)$ on $\mathcal{P}$. Then

$$\omega_\pm(X^h, Y^h)_K = (y_1 + y_3)^{-1}(X_1Y_2 - X_2Y_1), \qquad \omega_\pm(X^h, W)_{K=0},$$
$$\omega_\pm(W, W') = h(I \circ U, U') \pm h(J \circ V, V'), \quad \text{where } W' = (U', V') \in \mathcal{V}_K.$$

Set $V = \sum_s v_s J_s$. Then we easily obtain that

$$\begin{aligned} 3d\omega_\pm(X^h, Y^h, W)_K &= -(v_1 + v_3)(y_1 + y_3)^{-2}((X_1Y_2 - X_2Y_1)) \\ &\quad + h(R(X, Y)I, I \circ U) \pm h(R(X, Y)J, J \circ V) \end{aligned}$$

in view of (9) and the fact that $[X^h, W]_K$ and $[Y^h, W]_K$ are vertical vectors. Moreover

$$\begin{aligned} h(R(X, Y)J, J \circ V) &= -\langle R(X, Y)J, J \circ V\rangle \\ &= 2(y_1 + y_3)[y_2(v_1 - v_3) + v_2(y_1 - y_3)][\eta_1(R(X, Y)e_1) + \eta_2(R(X, Y)e_2)]. \end{aligned}$$

Thus putting $y_1 = 2$, $y_2 = 0$, $y_3 = \sqrt{3}$, $U = 0$, $v_1 = \sqrt{3}$, $v_2 = 0$, $v_3 = 2$ we see that $d\omega_\pm(X^h, Y^h, W) \neq 0$. Therefore the structure $(g, J_\pm)$ is not Kählerian.

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